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RESEARCH ARTICLE

CIRCLING THE SQUARE WITH STRAIGHTEDGE & COMPASS IN EUCLIDEAN GEOMETRY

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UK

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ABSTRACT

There are three classical problems remaining from ancient Greek mathematics which are extremely influential in the development of Geometry. They are Trisecting An Angle, Squaring The Circle, and Doubling The Cube problems. I solve the Squaring The Circle problem, of which paper is published in the International Journal Of Mathematics Trends And Technology (Volume 69, June 2023). Upstream from this method of exact "Squaring The Circle", we can deduce, conversely/inversely, to get a new Mathematical challenge "CIRCLING THE SQUARE" with a straightedge & a compass in Euclidean Geometry. This study idea came from my exact solution "Squaring The Circle by Straightedge & compass in Euclidean Geometry", published by IJMTT in June 2023 at https://ijmttjournal.org/Volume-69/Issue-6/IJMTT-V69I6P506.pdf for this ancient Greek Geometry problem. In this research, I adopt the ANALYSIS method to prove the process of solving this new challenge problem, which has not existed in the Mathematics field till today. The process is an inverse/converse solution solving the ancient Greek Geometry challenge problem of "Squaring The Circle", using a straightedge & a compass. I hereby commit that this is my own personal research project.

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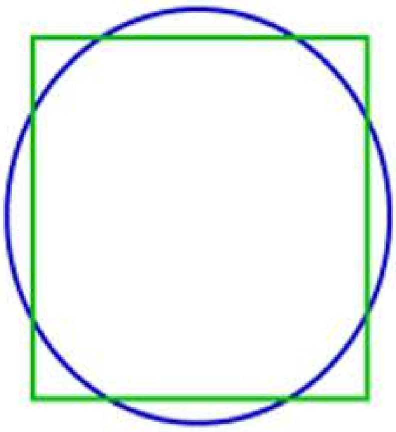
INTRODUCTION

'Doubling a cube', 'trisecting an angle', and 'squaring the circle' by straightedge & compass, are the problems first proposed in Greek mathematics, which were extremely influential in the development of Geometry. The history of the "Squaring The Circle" problem dates back millennia to around 450 B.C. (nearly 2,500 years), according to Quanta, a science and mathematics magazine. Mathematician Anaxagoras of Clazomenae was imprisoned for radical ideas about the sun, and while in prison, he worked on the now-iconic problem involving a compass and straightedge [1]. The present article studies what has become the most famous for these problems, namely the problem of squaring the circle or the quadrature of the circle, as it is sometimes called [2]. One of the fascinating aspects of this problem is that it has been of interest throughout the history of mathematics. From the oldest mathematical documents known up to today, the problems and related problems concerning π have interested both professional and non-professional mathematicians. In geometry, "straightedge and compass" construction is also known as Euclidean construction or classical construction [3]. Despite the proof of the impossibility of "squaring the circle," the problem has continued to capture the imaginations of mathematicians and the general public alike, and it remains an important topic in the history and philosophy of mathematics. In 2022, I solved the "Trisecting An Angle" problem with straightedge and compass [4], and published it in the IJMTT journal as a counter-proof to the Wantzel, L. proof in 1837 [5].

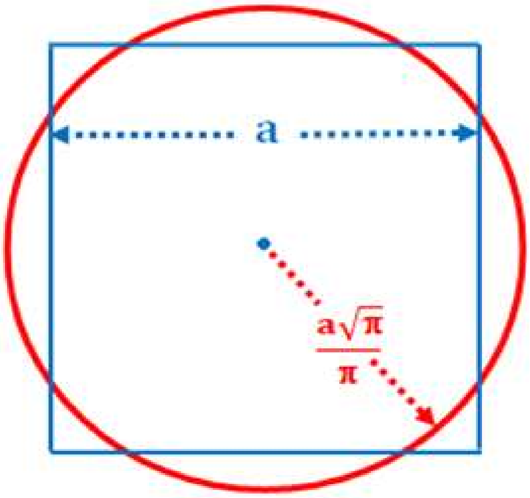


Although the above ancient Greek Maths Challenges are closely linked, I chose to solve the "Squaring The Circle" problem as my second research study after solving the "Trisecting An Angle" exactly and successfully. They were then published in the IJMTT, [4] and [6]. *In June 2023 I solved exactly and accurately the ancient Greek problem that has challenged mathematicians for over 2500 years - "Squaring The Circle" with a straightedge & compass. Then, the* International Journal Of Mathematics Trends And Technology *(IJMTT) published this paper on 23 June 2023* [6]. *After the paper was published, an idea derived from the solved result to create a new* *mathematical challenge, which had not existed before. The idea is as follows:*

"If we can square a given circle then how about to circle a given square, inversely/conversely". The following diagram illustrates this concept.

(inversely/conversely from one another)

In seeking the solutions to many mathematical problems, geometers developed a special technique, which they called "analysis". They assumed that the problem had been solved, and then, by investigating the properties of this solution, worked back to find an equivalent problem that could be solved based on the givens. To obtain a formally correct solution to the original problem, the geometers reversed the procedure. First, the data were used to solve the equivalent problem derived in the analysis, and from the solution obtained, the original problem was solved. In contrast to this analysis, this reversed procedure is called "synthesis". I adopted the technique "ANALYSIS" to solve accurately the "Squaring The Circle" problem with only a straightedge & compass [24]. I also used the technique "ANALYSIS" to solve the current challenge problem "CIRCLING THE SQUARE", derived from my "Squaring The Circle" solution mentioned above. *I hereby commit that this is my own personal research project.*



It is not saying that a circle of equal area to a square does not exist. If the square has an area equal to A, then a circle with a radius $r = \frac{\sqrt{\pi A}}{\pi}$ has the same area. Moreover, it is not saying that it is impossible, since it is possible, under the restriction of using only a straightedge and a compass [2].

This thesis paper includes a new geometrical shape defined as a "Conical-Arc," a proof of the intersection of an arbitrarily given square and a circle with the same area as the given square. In addition, proofs of some theorems show that the intersection of the square and the resulting circle is a regular octagon inscribed in the given square.

Lao Tzu (Author of Tao Te Ching):

"The great Tao is simple, very simple!"
(Lão Tử - Đạo Đức Kinh: Đại Đạo thì giản dị, rất giản dị !).

Proposition

Definition 1: "**Conical-Arc**" shape

Given a circle (O, r) and an angle $\widehat{BAC}$ with its vertex outside the circle such that the bisector of the angle passes through the centre O of the circle, then the special shape formed by the 2 sides of the angle and arc $\widehat{DE}$ can be called a Conical-Arc (in Figure 1 below, the red shape ADE is a Conical-Arc). If $\widehat{BAC}$ is a right angle then the shape ADE is called a Right-Conical-Arc.

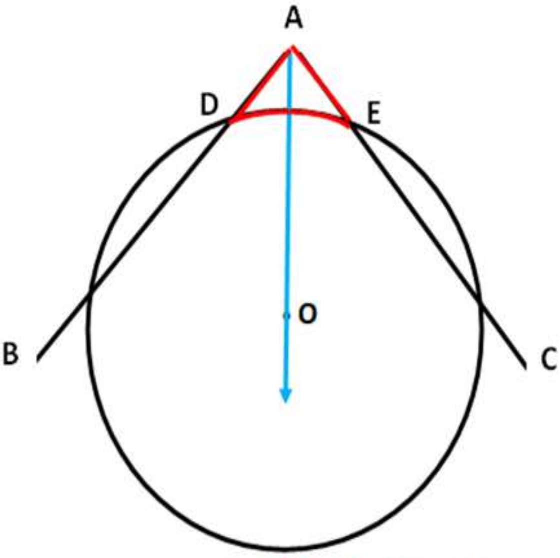


Figure 1: The Conical-Arc ADE

Theorem 1:

If there is a circle (O, r) of area $\pi r^2 = a^2$, of which the centre coincides with the centre of a given square ABCD (yellow colour in Figure 2 below), side a and area a^2,

then

a. *The circumference of the circle (O, r) intersects the square ABCD at 8 points a, b, c, d, e, f, g & h.*
b. *Square ABCD and circle (O, r) make 4 equal circle segments, attached to 4 sides AB, BC, CD & DA of the given square ABCD and located outside this square (in Figure 2 below).The result of this shows the areas of the 4 Right-Conical-Arcs are equal.*
c. *The areas of the four Right-Conical-Arcs formed by the four right angles $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$ and the four circle arcs ha, bc, de & fg of the circle (O, r) are equal to the areas of the four circle segments mentioned in section b. above.*

Proof

a. Consider circle (O, OA), which is the circumscribed circle of the given square ABCD (*orange colour in Figure 2 below*).

This circle occupied an area larger than area a^2 of the given square ABCD because area $\pi\frac{a^2}{2}$ of the circle is larger than area a^2 of the square.

Then, consider the inscribed circle (*blue colour in Figure 2 below*) of the square, which has an area $\pi\frac{a^2}{4}$ less than the area a^2 of the given square ABCD. And then,

$$\pi\frac{a^2}{4} < \mathbf{a^2} < \pi\frac{a^2}{2} \qquad \text{..............................(1)}$$

Thus, by (1) circumference of the concentric circle (O, r) with area a^2 is located in between the circumscribed circle (O, OA) and the inscribed circle (*blue colour*) of the square and intersects the 4 sides of the concentric square at 8 points a, b, c, d, e, f, g, h, as required (Figure 2 below).

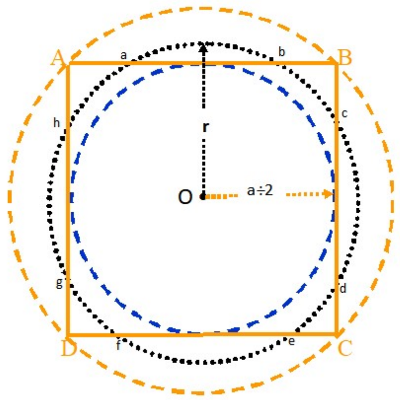


Figure 2. Four small segments (bordered by yellow line and black arc) locate above side AB, below side CD and left & right of sides AC & BD of the square ABCD.

a. In section a., the circle (O, r) cuts four sides AB, BC, CD, and DA of the given square at a, b, c, d, e, f, g, and h (*Figure 3 below*). Therefore, the intersection of the square and the black circle (O, r) shows us four circular segments, described in Figure 3 below, by the four regions limited between arcs ab, cd, ef, and gh and the four line segments (*red colour in Figure 3 below*).

- Because the circle (O, r) and square ABCD are concentric, the distances from the four line segments ab, cd, ef, and gh to the centre O are equal. This result shows that the above four circular segments are equal (Figure 3).
- This concentric property of the circle (O, r) and the square ABCD results in the four areas of the four Right-Conical-Arcs Aah, Bbc, Cde, and Dgf being equal (Figure 3 below).

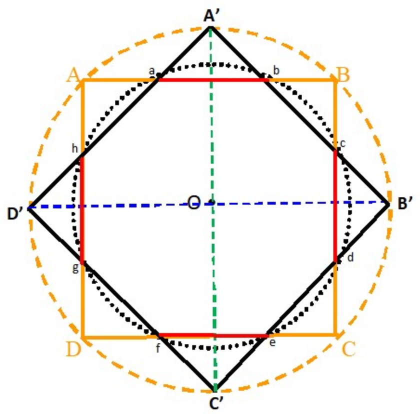


Figure 3. Four equal areas of the 4 circle segments, described by the 4 regions limited in between the arcs ab, cd, ef & gh (black colour) and the 4 line segments (red colour). And also 4 equal areas of the 4 Right-Conical-Arcs Aah, Bbc, Cde & Dgf.

b. *Aim to prove the areas of the four Right-Conical-Arcs formed by 4 right angles $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$ and 4 arcs of the circle (O, r) are equal to areas of the 4 segments in section b. above.*

Assume there exists a circle (O, r) that is concentric to the given square ABCD with side a, and has the same area as the area of the square $a^2 = \pi r^2$. Then, in section b., four circle segments, which are outside the square, are equal (Figure 4 below). These four circular segments were excluded from the intersection area of the square and circle (O, r). By the constraint of the assumption above {the area of the square a^2 = the area of the circle πr^2},

either *the intersection has to include these 4 circle segments to be equal to the area πr^2 of the circle (O, r),*
or *the intersection has to include these 4 Right-Conical-Arcs to be equal to the area a^2 of the square.*
Therefore, the areas of the four equal Right-Conical-Arcs *Aah, Bbc, Cde & Dgf* formed by the four right angles $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$ and the areas of the four equal circle segments are equal (Figure 4 below).

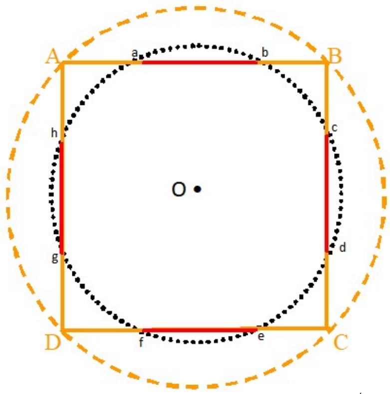


Figure 4. ***The intersection area of the square and the circle (O, r) needs EITHER the 4 areas of the 4 circle segments ab, cd, ef & gh to be equal to the area πr^2 of the circle (O, r) OR the 4 areas of the 4 right Conical-Arcs Aah, Bbc, Cde &Dfg to be equal to the area of the given square ABCD.***

Theorem 2: "ANALYSIS THEOREM"

Given a square ABCD with area a^2. If there exists a circle (O, r) with area $\pi r^2 = a^2$ (assumed), which is concentric with the square ABCD, then 4 sides of the square ABCD are overlapped 4 non-consecutive sides of a <u>regular octagon abcdefgh</u>, which is inscribed in the circle (O, r).

Proof

Assuming that there exists a circle (O, r) of area a^2 – *black colour in Figure 5 below*, which is concentric with the given square ABCD of area a^2, then by b. in section 2.2 of Theorem 1, the four circle segments, formed by the circle (O, r) at four sides of ABCD, are equal.

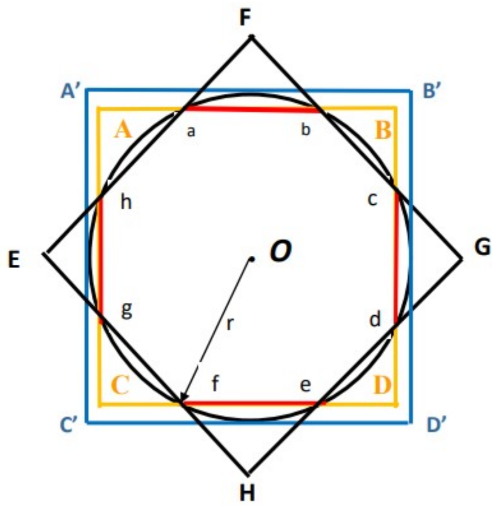


Figure 5. ***abcdefgh (red and black colours) is the regular Octagon inscribed in Circle (O, r).***

Consider the area of the Conical-Arc Aah in Figure 5 above (as defined in Definition 1) of vertex A of the square ABCD. From the expression {area a^2 of ABCD = area πr^2 of the circle (O, r)}, we obtain the following similar/equivalent expressions:

{the area of the Conical-Arc Aha = the area of the circle segment ab (red colour)} (2)

{the area of the Conical-Arc Bbc = the area of the circle segment cd} (3)

{the area of the Conical-Arc Dde = the area of the circle segment ef} (4)

{the area of the Conical-Arc Cfg = the area of the circle segment gh} (5)

Note that all expressions (2), (3), (4) & (5) above are illustrated in Figure 5 above and Figure 6 below.

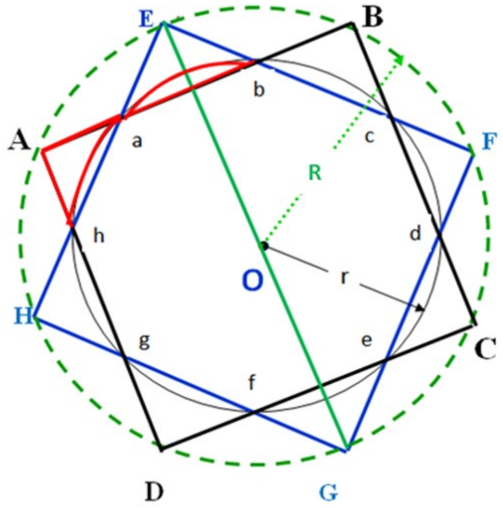


Figure 6. ***The inscribed squares ABCD (red and black colours) & EFGH (blue colour) of the circle (O, R).***

Let the given square ABCD, the circumscribed circle (O, R) – green colour - of this square ABCD, and the assumed circle (O, r) of area a^2 (black colour) all be concentric. Then, the extended arc chord ah of (O, r), in Figure 6, meets the circumscribed circle (O, R) – marked green dash in Figure 6 above – at E and H. Connect the diameter of (O, R) which gets through E and O. From E, draw a symmetric chord to EH that meets the green dashed circle (O, R) at F. In the special octagon abcdefgh, inscribed in the given circle (O, r) with four equal and parallel side pairs, section c. of Theorem 1 shows that EF is the symmetric chord of EH through the symmetric EO-axis (green colour). From section c. of Theorem 1 above, the distances between O and the two chords ha & bc are the same, and this equality shows that chord EF (blue colour in Figure 6 above) in the green dashed circle (O, R) overlaps chord bc of the given circle (O, r). Similarly, chord FG in the green dashed circle (O, R) also overlaps chord de (Figure 6 above) of the given black circle (O, r). By Section c. of Theorem 1, FG // EH, then chords EF and GH of the green dashed circle (O, R) are equal and parallel. This implies

EF = FG = GH = HE …………………(6)

and

EFGH (blue) is the inscribed square of the circle (O, R) ………..(7)

Then, (6) and (7) show that the areas of the two squares ABCD (black) & EFGH (blue) are the same and equal to a^2.

The locations of the eight sides of the equal squares ABCD and EFGH above show 8 chords ab, bc, cd, de, ef, fg, gh & ha of the given circle (O, r) are equal. Therefore, these eight equal chords show the shape abcdefgh is a regular octagon that inscribes in the assumed circle (O, r), as required.

Note that this proof also shows that the regular octagon abcdefgh in Figure 6 above has 4 sides overlapping the given square ABCD.

Theorem 3: ***The regular octagon abcdefgh, mentioned in the above Theorem 2 has an inscribed circle (O, $\frac{a}{2}$) that is also the inscribed circle of the given square ABCD.***

Proof: According to Theorem 2 above, the octagon abcdefgh has four sides attached to the given square ABCD (Figure 7 below). From the eight equal distances between the centre O and eight sides of the regular octagon abcdefgh, the inscribed circle (O, $\frac{a}{2}$) of this octagon (red colour in Figure 7 below) is also the inscribed circle of the given square ABCD. Thus, this circle had a radius of $\frac{a}{2}$ (Figure 7).

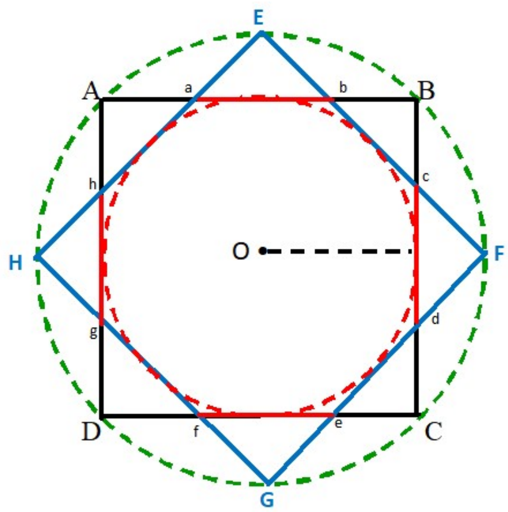


Figure 7. *The inscribed circle (O, $\frac{a}{2}$) – red colour - of the given squares ABCD (red and black colours).*

Theorem 4: "*Ruler Theorem*".

The circle (O, r) of area a^2, equalling the area a^2 of the given square ABCD of side a, and mentioned in the Analysis Theorem, creates the inscribed circle (O, $\frac{a}{2}$) of the square. This circle (O, $\frac{a}{2}$) is called the RULER of the "CIRCLING THE SQUARE" problem.

Proof: According to Theorems 2 and 3 in Sections 2.3 & 2.4, the blue circle (O, $\frac{a}{2}$) in Figure 8 below is the inscribed circle in both the given square ABCD and the regular octagon abcdefgh. This octagon is also an inscribed regular octagon of the resulting circle (O, r) of area $\pi r^2 = a^2$ and $r = a\frac{\sqrt{\pi}}{\pi}$.

Note that the resulting circle (O, r) of area $\pi r^2 = a^2$ equalling the area of the given square ABCD is unconstructed by straightedge & compass because its radius is the irrational number $r = a\frac{\sqrt{\pi}}{\pi}$.

However, this RULER THEOREM shows us a RULER of the "CIRCLING THE SQUARE" problem, which can be used to construct the resulting circle (O, $r = a\frac{\sqrt{\pi}}{\pi}$) with a straightedge & compass, as this RULER of the "CIRCLING THE SQUARE" problem is proved a circle (O, $\frac{a}{2}$) having radius $\frac{a}{2}$ = ½ side of the given square of the "CIRCLING THE SQUARE" problem.

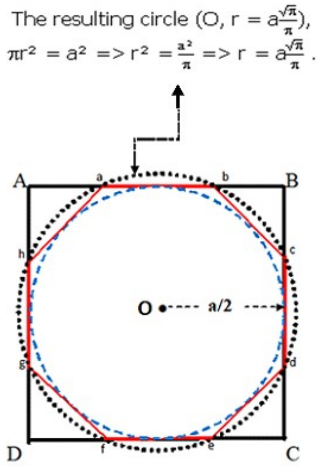


Figure 8. ***The blue circle (O, $\frac{a}{2}$) that inscribes in both the given squares ABCD (red and black colours) and the octagon abcdefgh (red colour) is the RULER OF THE "CIRCLING THE SQUARE" problem, mentioned in Theorem 4 above.***

Theorem 5: ***Solution for "Circling the Square"***

It is possible to construct a circle, which has the same area as the area a^2 of a given square ABCD (side a) with a straightedge and a compass.

Proof: Given a square ABCD of side a, $a \subset \mathbb{R}$ and area a^2, then by the Definition 1 and the Theorems 1, 2, 3 & 4 above, we have the Ruler of the "Circling The Square" problem, which is the inscribed circle (O, $\frac{a}{2}$) of both the square and the inscribed regular octagon of the square. This circle (O, $\frac{a}{2}$) can be constructed using a straightedge and a compass.

Meanwhile, the circumscribed regular octagon abcdefgh of the circle (O, $\frac{a}{2}$) is also constructed by a straightedge and compass (Figure 9 below). Finally, the circumscribed circle (O, r) with area $\pi r^2 = a^2$ and $r = a\frac{\sqrt{\pi}}{\pi}$ of the octagon is constructed by a straightedge and compass, as its radius r is the constructive distance from the vertex a/b/c/d/e/f/g or h to the centre O (Figure 9).

Construction Solution: From the above sections 2.1, 2.2, 2.3, 2.4, 2.5, and 2.6, the construction of the exact solution for the "CIRCLING THE SQUARE" problem with a straightedge and compass is as follows:

a. For the given square ABCD side a, use a straightedge and a compass to draw two line segments that divide the square into four equal parts. Then, draw its inscribed blue circle (O, $\frac{a}{2}$) and its diagonals AC and BD (the 1st image in Figure 9 below).

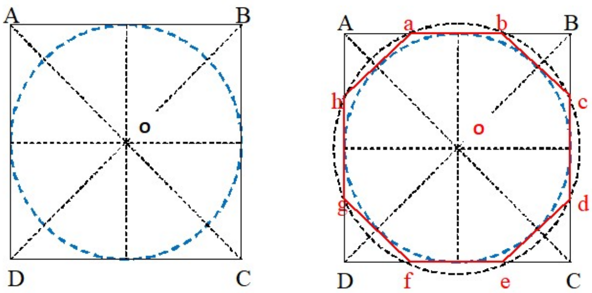


Figure 9. ***The image illustrates the constructive solution of the "CIRCLING THE SQUARE" problem.***

b. Use a straightedge and compass to draw the regular octagon abcdefgh, which circumscribes the circle (O, $\frac{a}{2}$) by NOTE that this octagon has 4 sides perpendicular to the 2 diagonals of square ABCD and the other 4 sides overlaps the 4 sides of ABCD. Then, in the octagon, use the distance from any of its vertexes a, b, c, d, e, f, g and h to the centre O to construct the circle (O, r), area $\pi r^2 = a^2$, and $r = a\frac{\sqrt{\pi}}{\pi}$ (*the 2nd image of Figure 9 above*).

This circle (O, r), area $\pi r^2 = a^2$ and $r = a\frac{\sqrt{\pi}}{\pi}$, constructed with only straightedge & compass, is the solution for the "CIRCLING THE CIRCLE" problem.

DISCUSSION AND CONCLUSON

Can mathematicians use a compass and a straightedge to construct a circle having an area equal to a given square exactly/accurately?

This question is the same as for constructing the mentioned circle or finding an accurate solution for the new challenge Mathematics problem "CIRCLING THE SQUARE". Surprisingly, only I, myself, was still working on this question because the challenge problem had just arisen contemporarily when I ended up my original research article *"Exact Squaring The Circle with Straightedge and Compass by Secondary Geometry"*, published by IJMTT in June 2023 [6]. In 2017, Andras Máthé and Oleg Pikhurko of the University of Warwick and Jonathan Noel of the University of Victoria were the latest authors who joined this ancient tradition challenge (Squaring The Circle problem). These authors showed how a circle can be squared by cutting it into pieces that can be visualized and drawn. This result builds on a rich history. Mathematicians named this method "the equidecomposition" but it is also theoretical proof that the problem can be solved (without a straightedge & compass) by cutting the circle into pieces and rearranging it into a square and none knows the number of pieces. Nevertheless, no computers existed in the ancient Greek era [25]. *In June 2023, I solved exactly and accurately the ancient Greek problem that has challenged mathematicians for over 2500 years - "Squaring The Circle" with a straightedge & compass.*

Then, the International Journal Of Mathematics Trends And Technology (*IJMTT) published this paper on 23 June 2023 [6].*

After the paper was published, an idea derived from the solution to create a new mathematical challenge, which had not existed before. The idea is as follows:

"If we can square a given circle then how about to circle a given square, inversely/conversely". The following diagram illustrates this concept idea:

"*Squaring The Circle*"  "*Circling The Square*"
(inversely/conversely from one another)

The "Analysis" method is applied correctly to Geometry to complete this research study to gain an exact/accurate solution to this new challenge "CIRCLING THE SQUARE" problem in Mathematics.

The results of my independent research show that the correct answer (O, r), constructed by compass and straightedge, has the exact area $a^2 = \pi r^2$, therefore if the given square to circle is a unit square $a^2 = 1$, then in terms of geometry, π can be constructive/expressed by a circle with area $\pi r^2 = 1$. This circle comes from the *RULER of the "CIRCLING THE SQUARE" problem* yielded by the given unit square. Subsequently, the exact geometric length of $\pi = \frac{1}{r^2}$ was determined. In practice, if the *International Bureau of Weights and Measures (BIPM),* the *International System of Units,* or any accurate laser measurement is used to measure the arithmetic value r of the answer circle (O, r), we can use this r to measure as accurately as possible to obtain the arithmetic value of $\pi = \frac{1}{r^2}$.

The above arithmetic value of π could be the nearest arithmetical value of the irrational number π ever seen.

My construction method is quite different from approximation and is based on using a straightedge and compass within secondary Geometry so that any secondary student can solve the problem for any given square. Moreover, this method shows that the value $r = \frac{\sqrt{\pi}}{\pi}$ can be expressed accurately, and the value $\pi = \frac{1}{r^2}$ or $\pi = \frac{a^2}{r^2}$, $a \subset \mathbb{R}$ can also be expressed accurately in terms of Geometry. This Geometrical expression of the irrational number π could be an interesting field for mathematicians in the 21st century. In other words, algebraic geometry can express exactly any irrational number kπ, $k \subset \mathbb{R}$.

In addition, this research result can be used for further research in the “SPHERING THE CUBE” challenge, with only “a straightedge & a compass” in Euclidean Geometry.

Furthermore, the research also opens some new challenge problems which can be “CIRCLING THE EQUILATERAL TRIANGLE”, “CIRCLING THE REGULAR PENTAGON”, “CIRCLING THE REGULAR HEXAGON”, “CIRCLING THE REGULAR OCTAGON” etc …, using a straightedge and compass.

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