International Journal of Recent Advances in Multidisciplinary Research
Vol. 02, Issue 08, pp.0670-0676, August, 2015

Review Article

PARTIALLY ORDERED RADICALS IN PO-TERNARY SEMIRING

[1]Siva Prasad, P., [2]Dr. MadhusudhanaRao, D. and [3]Srinivasa Rao, G.

[1]Department of Mathematics, Universal College of Engineering & Technology, Perecherla, Guntur, A. P. India
[2]Department of Mathematics, V.S.R. and N.V.R. College, Tenali, A. P. India
[3]Department of Mathematics, Tirumala Engineering College, Narasaraopet, A. P. India

ARTICLE INFO

Article History:
Received 12th May 2015
Received in revised form
30th June, 2015
Accepted 21st July, 2015
Published online 31st August, 2015

Keywords:
Primepo-Radical,
Completely Prime Po-Radical,
Prime,
Semiprime,
Completely Semiprime.

ABSTRACT

In this paper we made a study on prime po-radicals in partially ordered ternary semiring and characterized these ideals. Mathematics Subject Classification: 16Y30, 16Y99.

INTRODUCTION

The notion of semiring was introduced by Vandiver [6] in 1934. In fact semiring is a generalization of ring. In 1971 Lister [4] characterized those additive subgroups of rings which are closed under the triple ring product and he called this algebraic system a ternary ring.T. K. Dutta and S. Kar initiated prime ideals and prime radical of ternary semir-ings in [1]. The same researchers launched semiprime ideals and irreducible ideals of ternary semirings[2]. Furthermore S. Kar came up with the notion of quasi-ideals and bi-ideals in ternary semirings. Similarly, M. Shabir and Bashir.S prime bi-ideals in ternary semigroups in [5]. we assemble requisite material on partially ordered radicals in partially ordered ternary semirings.

Preliminaries

In this section, the required preliminaries are presented.

Definition 2.1 : A nonempty set T together with a binary operation called addition and a ternary multiplication denoted by [ ] is said to be a *ternary semiring* if T is an additive commutative semigroup satisfying the following conditions:

i) $[[abc]de] = [a[bcd]e] = [ab[cde]]$,
ii) $[(a + b)cd] = [acd] + [bcd]$,
iii) $[a(b + c)d] = [abd] + [acd]$,
iv) $[ab(c + d)] = [abc] + [abd]$ for all $a; b; c; d; e \in$T.

Note 2.2 : For the convenience we write $x_1x_2x_3$ instead of $\left[x_1x_2x_3\right]$

***Corresponding author: Dr. MadhusudhanaRao, D. ,**
Department of Mathematics, V.S.R. and N.V.R. College, Tenali, A. P. India

Note 2.3: Let T be a ternary semiring. If A, B and C are three subsets of T, we shall denote the set
ABC = $\{\Sigma abc : a \in A, b \in B, c \in C\}$.

Note 2.4: Let T be a ternary semiring. If A, B are two subsets of T, we shall denote the set A + B = $\{a+b : a \in A, b \in B\}$ and 2A = $\{a + a: a \in A\}$.

Note 2.5: Any semiring can be reduced to a ternary semiring.

Definition 2.6: A ternary semiring T is said to be a *partially ordered ternary semiring* or simply *PO Ternary Semiring*or*Ordered Ternary Semiring*provided T is partially ordered set such that $a \leq b$ then

(1) $a + c \leq b + c$ and $c + a \leq c + b$,
(2) $acd \leq bcd$, $cad \leq cbd$ and $cda \leq cdb$ for all $a, b, c, d \in$ T.

Throughout Twill denote as PO-ternary semiring unless otherwise stated.

Note 2.7: Some times we write $a \geq b$ for $a \leq b$. That is "$\geq$" is the dual relation of "$\leq$".

Theorem 2.8: Let T be a po-ternary semiring and A ⊆T, B ⊆T and C ⊆ T. Then (i) A ⊆(A], (ii) ((A]] = (A], (iii) (A](B](C] ⊆(ABC] and (iv) A ⊆B ⇒A ⊆(B] and (v) A ⊆B ⇒(A] ⊆(B], (vi) (A ∩ B] = (A] ∩ (B], (vii) (A ∪ B] = (A] ∪ (B].

Definition 2.9: A nonempty subset A of a PO-ternary semiring T is a *PO-ternary ideal* of T provided A is additive subsemigroup of T, ATT ⊆ A, TTA ⊆ A, TAT ⊆ A and (A] ⊆ A.

Definition 2.10: A PO-ternary ideal A of a PO-ternary semiring T is said to be a *completely prime PO-ternary ideal* of T provided $x, y, z \in$ T and $xyz \in$ A implies either $x \in$ A or $y \in$ A or $z \in$ A.

Definition 2.11: Let T be a ternary semiring. A nonempty subset A of T is said to be a *PO-c-system* of T if for each a, b, $c \in$ A there exist an element $d \in$ A such that $d \leq abc$.

Definition 2.12: A nonempty subset A of a PO-ternary semiring T is said to be a *PO-m*-system provided for any $a, b, c \in$ A there exist $d \in$ A and $x, y \in$ T such that $d \leq axbyc$.

Definition 2.13: Let T be a PO-ternary semiring. A non-empty subset A of T is said to be a *PO-d*-system of T if for each $a \in$ A there exist an element $c \in$ A such that $c \leq a^n$ for all odd natural number n.

Definition 2.14: A non-empty subset A of a PO-ternary semiring T is said to be a *PO-n*-system provided for any $a \in$ A there exist $d \in$ A and $x, y \in$ T such that $d \leq axaya$.

Theorem 2.15: A PO-ternary ideal A of a PO-ternary semiring T is completely prime if and only if T\A is either PO-*c-system* of T or empty.

Theorem 2.16: Every completely prime PO-ternary ideal of a PO-ternary semiring T is a prime PO-ternary ideal of T.

Theorem 2.17: Let T be a commutative PO-ternary semiring. A PO-ternary ideal P of T is a prime PO-ternary ideal if and only if P is a completely prime PO-ternary ideal.

Theorem 2.18: If T is a globally idempotent PO-ternary semiring then every maximal PO-ternary ideal of T is a prime PO-ternary ideal of T.

Definition 2.19: A PO-ternary ideal A of a PO-ternary semiring T is said to be a *prime PO-ternary ideal* of T provided X,Y,Z are PO-ternary ideals of T and XYZ ⊆ A ⟹ X ⊆A or Y ⊆A or Z ⊆A.

Definition 2.20: A PO-ternary ideal A of a PO-ternary semiring T is said to be a *completely Semiprime PO-ternary ideal* provided $x \in$ T, $x^n \in$ A for some odd natural number n>1 implies $x \in$ A.

Definition 2.21: A PO-ternary ideal A of a PO-ternary semiring T is said to be *SemiprimePO-ternary ideal* provided X is a PO-ternary ideal of T and $X^n \subseteq$ A for some odd natural number $n$impliesX ⊆ A.

Theorem 2.22: Every completely prime PO-ternary ideal of a PO-ternary semiring T is a completely Semiprime PO-ternary ideal of T.

Theorem 2.23: Let T be a commutative PO-ternary semiring. A PO-ternary ideal A of T is completely semiprime if and only if it is semiprime.

Theorem 2.24: Every PO-*m*-system in a PO-ternary semiring T is a PO-*n*-system.

Theorem 2.25: A PO-ternary ideal Q of a PO-ternary semiring T is a semiprime PO-ternary ideal if and only if T\Q is a PO-*n*-system of T or empty.

PrimePO-Radicaland Completely PrimePO-Radical

We use the following notation.

Notation 3.1: If A is a PO-ternary ideal of a PO-ternary semiring T, then we associate the following four types of sets.

A_1 = The intersection of all completely prime PO-ternary ideals of T containing A.

$A_2 = \{x \in T : x^n \in A$ for some odd natural numbers $n\}$

A_3 = The intersection of all prime PO-ternary ideals of T containing A.

$A_4 = \{x \in T : < x >^n \subseteq$ A for some odd natural number $n\}$

Theorem 3.2: If A is aPO-ternary ideal of a PO-ternary semiring T, then
A$\subseteq A_4 \subseteq A_3 \subseteq A_2 \subseteq A_1$.

Proof:

i) A $\subseteq A_4$: Let $x \in$ A . Then $<x> \subseteq$ A and hence $x \in A_4$. Therefore A $\subseteq A_4$

ii) $A_4 \subseteq A_3$: Let $x \in A_4$. Then $< x >^n \subseteq$ A for some odd natural number *n.*

Let P be any prime ideal of T containing A.

Then $< x >^n \subseteq$ A for some odd natural number $n \Rightarrow < x >^n \subseteq$ P.

Since P is prime ,$<x> \subseteq$ P and hence $x \in$ P.

Since this is true for all prime PO-ternary ideals of P containing A, $x \in A_3$.

Therefore $A_4 \subseteq A_3$

iii) $A_3 \subseteq A_2$: Let $x \in A_3$. Suppose if possible $x \notin A_2$.

Then $x^n \notin$A for all odd natural number *n*.

Consider Q = $\bigcup x^n$ for all odd natural number *n*, and $x \in$ T.

Let *a, b, c* $\in$Q. Then *a*= $(x)^r$, *b* = $(x)^s$, *c* = $(x)^t$ for some odd natural numbers *r, s, t*.

Therefore *abc*= $(x)^r\ (x)^s\ (x)^t = x^{r+s+t} \in Q$ and hence Q is a PO-*c*-system of T.

By theorem 2.15, P = T\Q is a completely prime PO-ternary ideal of T and $x \notin P$.

By theorem 2.16, P is a prime PO-ternary ideal of T and $x \notin P$. Therefore $x \notin A_3$.

It is a contradiction. Therefore $x \in A_2$ and hence $A_3 \subseteq A_2$.

iv) $A_2 \subseteq A_1$: Let $x \in A_2$. Now x $\in A_2 \Rightarrow x^n \in$ A for some odd natural number *n*.

Let P be any completely prime PO-ternary ideal of T containing A.

Then $x^n \in$ A$\subseteq$P $\Rightarrow x^n \in$ P $\Rightarrow x \in$ P.Therefore $x \in A_1$ .Therefore $A_2 \subseteq A_1$.

Hence A $\subseteq A_4 \subseteq A_3 \subseteq A_2 \subseteq A_1$.

Corollary 3.3: If A is aPO-ternary ideal of a PO-ternary semiring T, then (A] $\subseteq$ (A_4] $\subseteq$ (A_3] $\subseteq$ (A_2] $\subseteq$ (A_1].

Proof **:** It is easy to verify the proof by theorem 2.8 and by theorem 3.2.

Theorem 3.4: If A is aPO-ternary ideal of a commutative PO-ternary semiring T, then $A_1 = A_2 = A_3 = A_4$

Proof : By theorem 3.2, A $\subseteq A_4 \subseteq A_3 \subseteq A_2 \subseteq A_1$. By theorem 2.17, in a commutative PO-ternary semiring T, an ideal A is a prime PO-ternary ideal if A is completely prime PO-ternary ideal. So $A_1 = A_3$. By theorem 2.23, in a commutative PO-ternary semiring T aPO-ternary ideal A is semiprime if and only if A is completely semiprime PO-ternary ideal.
So $A_4 = A_2$ and hence $A_1 = A_2 = A_3 = A_4$ **.**

Corollary 3.5: If A is aPO-ternary ideal of a commutative PO-ternary semiring T, then (A_1] = (A_2] = (A_3] = (A_4].

Proof**:** It is easy to verify the proof by theorem 2.8 and by theorem 3.4.

Note 3.6: In an arbitrary PO-ternary semiring (A_1 **]** $\neq$ **(** A_2 **]** $\neq$ **(** A_3 **]** $\neq$ **(** A_4 **].**

Example 3.7: Let T be the free ternary semigroup generated by *a*, *b*, *c*.

It is clear that A = (T a^3 T] is aPO-ternary ideal of T. Since $a^5 \in$ (T a^3 T], we have a $\in$ (A_2].

Evidently $(abc)^n \notin$ (T a^3 T] for all odd natural numbers *n* and thus abc $\notin$ (A_2].

Thus (A_2]is not aPO-ternary ideal of T. Therefore (A_1] $\neq$ (A_2] and (A_2] $\neq$ (A_3 **].**

We now introduce prime PO-radical and complete prime PO-radical of a PO-ternary ideal in a PO-ternary semiring.

Definition 3.8 : If A is a PO-ternary ideal of a PO-ternary semiring T, then the intersection of all prime PO-ternary ideals of T containing A is called *prime PO-radical* or simply PO-*radical* of A and it is denoted by $\sqrt{A}$ or *rad* A.

Definition 3.9: If A is a PO-ternary ideal of a PO-ternary semiring T , then the intersection of all completely prime PO-ternary ideals of T containing A is called *completely prime PO-radical* or simply *complete PO-radical* of A and it is denoted by *c.rad* A.
Note 3.10: If A is a PO-ternary ideal of a PO-ternary semiring T, then *rad* A = A_3, *c.rad* A = A_1 and *rad* A ⊆*c.rad* A**.**

Note 3.11: If (A] is a PO-ternary ideal of a PO-ternary semiring T, then *rad*(A] = (A_3], *c.rad*(A] = (A_1 **]**and *rad* (A]⊆*c.rad* (A]**.**

Corollary 3.12: If $a \in \sqrt{(A]}$, then there exist aodd positive integer *n* such that $a^n \in$ (A].
Proof:
By corollary 3.3, (A_3] $\subseteq$ (A_2] and hence $a \in \sqrt{(A]}$ = (A_3] $\subseteq$ (A_2]. Therefore $a^n \in$ (A] for some odd positive integer *n*.

Corollary 3.13: If A is aPO-ternary ideal of a commutative PO-ternary semiring T, then *rad*(A] = *c.rad*(A].

proof **:** By corollary 3.5,*rad*(A] = *c.rad*(A].

Corollary 3.14: If A is a PO-ternary ideal of a PO-ternary semiring T then *c.rad*(A] is a completely SemiprimePO-ternary ideal of T.
proof :

By theorem 2.22, *c.rad* (A] is a completely semiprimePO-ternary ideal of T.

Theorem 3.15: If A, B and C are any three PO-ternary ideals of a PO-ternary semiring T , then

- **A ⊆ B** $\Rightarrow \sqrt{A} \subseteq \sqrt{B}$
- **if A ∩ B∩ C ≠ ∅ then** $\sqrt{ABC} = \sqrt{A \cap B \cap C} = \sqrt{A} \cap \sqrt{B} \cap \sqrt{C}$
- **iii)** $\sqrt{\sqrt{A}} = \sqrt{A}$ **.**

- $\sqrt{A+B} = \sqrt{\sqrt{A}+\sqrt{B}}$

Proof :

- Suppose that $A \subseteq B$. If P is a prime PO-ternary ideal containing B then P is a prime PO-ternary ideal containing A. Therefore $\sqrt{A} \subseteq \sqrt{B}$.
- Let P be a prime PO-ternary ideal containing ABC.

Then ABC ⊆ P ⇒A⊆ P or B ⊆ P or C ⊆ P

⇒ $A \cap B \cap C$ ⊆P.Therefore P is a prime PO-ternary ideal containing A ⋂ B ⋂ C.
Therefore*rad* ($A \cap B \cap C$) ⊆ *rad*(ABC).Now let P be a prime PO-ternary ideal containing $A \cap B \cap C$. Then $A \cap B \cap C$ ⊆ P ⇒ABC ⊆ $A \cap B \cap C$ ⊆ P⇒ ABC ⊆ P.

Hence P is a prime PO-ternary ideal containing ABC.
Therefore *rad* (ABC) ⊆*rad*($A \cap B \cap C$). Hence*rad*(ABC) = *rad*($A \cap B \cap C$).

Since $A \cap B \cap C \neq \emptyset$, it is clear that A ∩ B ∩ C is aPO-ternary ideal in T.

Let $x \in \sqrt{A \cap B \cap C}$.
Then there exists an odd natural number $n \in$ N such that $x^n \in A \cap B \cap C$.

Therefore $x^n \in$ A, $x^n \in$ B and $x^n \in$ C. It follows that $x \in \sqrt{A}$, $x \in \sqrt{B}$ and $x \in \sqrt{C}$. Therefore $x \in \sqrt{A} \cap \sqrt{B} \cap \sqrt{C}$. Consequently, $x \in \sqrt{A} \cap \sqrt{B} \cap \sqrt{C}$ implies that there exists odd natural numbers *n*, *m, p*∈ N such that $x^n \in$ A, $x^m \in$ B and $x^p \in$ C. Clearly, $x^{nmp} \in$A ∩ B ∩ C. Thus $x \in \sqrt{A \cap B \cap C}$.

Therefore if A ∩ B ∩ C ≠ ∅then $\sqrt{A \cap B \cap C} = \sqrt{A} \cap \sqrt{B} \cap \sqrt{C}$.

iii) $\sqrt{A}$ = The intersection of all prime ideals of T containing A.

Now $\sqrt{\sqrt{A}}$ = The intersection of all prime PO-ternary ideals of T containing $\sqrt{A}$.
= The intersection of all prime PO-ternary ideals of T containing A = $\sqrt{A}$

Therefore $\sqrt{\sqrt{A}} = \sqrt{A}$.

iv) By condition i) we have A + B $\subseteq \sqrt{A}+\sqrt{B}$ and so $\sqrt{A+B} \subseteq \sqrt{\sqrt{A}+\sqrt{B}}$. Also by condition i) we have $\sqrt{A}+\sqrt{B} \subseteq \sqrt{A+B}$ and hence by using condition ii), $\sqrt{\sqrt{A}+\sqrt{B}} \subseteq \sqrt{\sqrt{A+B}} = \sqrt{A+B}$. Therefore $\sqrt{A+B} = \sqrt{\sqrt{A}+\sqrt{B}}$.

Corollary 3.16: If A, B and C are any three PO-ternary ideals of a PO-ternary semiring T, then

i) $A \subseteq B \Rightarrow \sqrt{(A]} \subseteq \sqrt{(B]}$
ii) if A ∩ B∩ C ≠ ∅ then $\sqrt{(A](B](C]} = \sqrt{(A] \cap (B] \cap (C]} = \sqrt{(A]} \cap \sqrt{(B]} \cap \sqrt{(C]}$
iii) $\sqrt{\sqrt{(A]}} = \sqrt{(A]}$.
iv) $\sqrt{(A+B]} = \sqrt{\sqrt{(A]}+\sqrt{(B]}}$

Theorem 3.17: If A is a PO-ternary ideal of a PO-ternary semiring T then $\sqrt{(A]}$ **is a semiprime ideal of T.**

proof : By theorem 2.23 , $\sqrt{(A]}$ is a semiprime ideal of T.

Theorem 3.18: A PO-ternary ideal Q of PO-ternary semiring T is a semiprimePO-ternary ideal of T if and only if $\sqrt{Q}$ =Q.

Proof : Suppose that Q is a semiprimePO-ternary ideal. Clearly Q ⊆ √Q.
Suppose if possible √Q ⊈ Q.Let a∈ √Q and a∉ Q.

Now a∉ Q ⇒a∈ T\Q and Q is semiprime. By theorem 2.25, T\Q is aPO-n-system.
By theorem 2.24, there exists aPO-m-system M such that a∈ M ⊆ T\Q.

Q ⊆ T\M and now T\M is a prime PO-ternary ideal of T, $a \notin$T\M.
It is a contradiction.Therefore √Q ⊆ Q. Hence √Q = Q.

Conversely suppose that Q is aPO-ternary ideal of T such that √Q = Q.
By corollary 3.17, √Q is a semiprimePO-ternary ideal of T. Therefore Q is semiprime.

Corollary 3.19: A PO-ternary ideal Q of a PO-ternary semiring T is a semiprimePO-ternary ideal if and only if Q is the intersection of all prime PO-ternary ideal of T contains Q.

Proof **:** By theorem 3.18., Q is semiprimeiff Q is the intersection of all prime PO-ternary ideals of T contains Q.

Corollary 3.20: If A is aPO-ternary ideal of a PO-ternary semiring T, then √A is the smallest semiprimePO-ternary ideal of T containing A.

Proof : We have that √A is the intersection of all prime PO-ternary ideals containing A in T.

Since intersection of prime PO-ternary ideals is semiprime, we have √A is semiprime.
Further, let Q be any semiprimePO-ternary ideal containing A, i.e. A⊆ Q. So √A ⊆ √Q.

Since Q is semiprime, By theorem 3.18, √Q = Q. Therefore √A ⊆ Q.
Hence √A is the smallest semiprimePO-ternary ideal of T containing A.

Theorem 3.21: If P is a prime PO-ternary ideal of a PO-ternary semiring T, then $\sqrt{(P)^n} = P$ for all odd natural numbers $n \in$ N.

Proof **:**We use induction on n to prove $\sqrt{P^n} = P$.
First we prove that $\sqrt{P} = P$. Since P is a prime PO-ternary ideal, $P \subseteq \sqrt{P} \subseteq P \Rightarrow \sqrt{P} = P$.

Assume that $\sqrt{P^k} = P$ for odd natural number k such that $1 \le k < n$.

Now $\sqrt{P^{k+2}} = \sqrt{P^k.P.P} = \sqrt{P^k} \cap \sqrt{P} \cap \sqrt{P} = \sqrt{P} \cap \sqrt{P} \cap \sqrt{P} = \sqrt{P} = P$.

Therefore $\sqrt{P^{k+2}} = P$. By induction $\sqrt{P^n}$ = P for all odd natural number $n \in$ N.

Theorem 3.22: In a PO-ternary semiring T with identity there is a unique maximal PO-ternary ideal M such that $\sqrt{(M)^n}$ = M for all odd natural numbers n $\in$ N.

Proof: Since T contains identity, T is a globally idempotent PO-ternary semiring.
Since M is a maximal PO-ternary ideal of T, by theorem 2.24 M is prime.
By theorem 3.21, $\sqrt{(M)^n}$ = M for all odd natural numbers n.

Theorem 3.23: If A is a PO-ternary ideal of a PO-ternary semiring T then $\sqrt{A}$ ={x ∈ T : every m-system of T containing x meets A } i.e., $\sqrt{A} = \{ x \in T : M(x) \cap A \neq \varnothing \}$.

Proof: Suppose that $x \in \sqrt{A}$. Let M be aPO-m-system containing x.

Then T\M is a prime PO-ternary ideal of T and $x \notin$ T\M. If M ⋂ A = ∅ then A ⊆ T\M.

Since T\M is a prime PO-ternary ideal containing A, $\sqrt{A} \subseteq$ T\M and hence $x \in$ T\M.

It is a contradiction. Therefore M(x)$\bigcap$ A $\neq \emptyset$. Hence $x \in \{x \in T : M(x) \bigcap A \neq \emptyset\}$. Conversely suppose that $x \in \{x \in T : M(x) \bigcap A \neq \emptyset\}$.

Suppose if possible $x \notin \sqrt{A}$. Then there exists a prime PO-ternary ideal P containing A such that $x \notin$ P. Now T\P is aPO-*m*-system and $x \in$ T\P.

A $\subseteq$ P $\Rightarrow$ T\P$\bigcap$ A = $\emptyset \Rightarrow x \notin \{x \in T : M(x) \bigcap A \neq \emptyset\}$.

It is a contradiction. Therefore $x \in \sqrt{A}$. Thus $\sqrt{A} = \{x \in T : M(x) \bigcap A \neq \emptyset\}$.

Conclusion: In this paper mainly we studied about prime, semiprimePO-ternary ideal in PO-ternary semiring.

Acknowledgments

Our thanks to the experts who have contributed towards preparation and development of the paper.

REFERENCES

Dutta, T.K. and Kar, S. 2003. On regular ternary semirings, Advances in Algebra, Proceedings of the ICM Satellite Conference in Algebra and RelatedTopics, World Scientific, New Jersey, 343-355.
Dutta, T.K. and Kar, S. 2006. A note on regular ternary semirings, Kyung-pook Math. J., IV6 , 357-365.
Kar, S. 2005. On quasi-ideals and bi- ideals in ternary semirings, *Int. J. Math.Math.Sc.,* 18, 3015-3023.
Lister, W.G. 1971. Ternary Rings, *Amer. Math. Soc.,* 154, 37-55.
Shabir, M. and Bashir, S. 2009. Prime ideals in ternary semigroups, *Asian European J. Math*., 2, 132-139.
Vandiver, 1934. Note on Simple type of algebra in which the cancellation law of addition does not hold, Bull.Am.Math.Soc.40, 920.

****