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RESEARCH ARTICLE

COINCIDENCE AND FIXED POINTS VIA MULTIVALUED MAPPINGS ON PARTIAL METRIC SPACES

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ABSTRACT

In this paper, we prove coincidence and common fixed point theorems for two pair of multivalued and single valued mappings in complete partial metric spaces. These results improve, extend and generalize the corresponding results in the literature. Moreover, an example is provided to illustrate the usability of main results.

INTRODUCTION

Nadler [10] in 1969 introduced the multivalued version of Banach contraction principle. Therefore metric fixed point theory of single valued mappings has been extended to multivalued mappings. The concept of partial metric spaces as a generalization of metric spaces was initiated in 1994 by Mathews [11], in his treatment of denotational semantics of dataflow networks and he proved the Banach contraction principle in such spaces. Many authors followed his ideas and proved various results, especially in fixed point theory. For partial metric spaces, self distance need not be zero. It was shown that, sometime the results of fixed point in partial metric spaces can be obtained directly from their induced metric counterparts [2, 5, 7, 8]. However, some conclusions important for the application of partial metrics in information sciences cannot be obtained in this way. For example, if x is a fixed point of mapping f, then from the method given in [8], we cannot conclude that $p(fx, fx) = 0 = p(x, x)$. In 2012, Aydi et al. [3] introduced the concept of a partial Hausdorff metric. They initiated study of fixed point theory for multivalued mappings on partial metric space using the partial Hausdorff metric and proved an analogue of well known Nadler fixed point theorem. Our proved results generalize the results of R. Damjanovic, B. Samet and C. Vetro [9]. Before presenting our main results, we start with introducing some definitions.

Definition 1.1: Let X and Y be non empty sets. T is said to be a multivalued mapping from X to Y if T is a function from X to the power set of Y. We denote a multivalued mapping by $X \longrightarrow 2^y$.

A point $x \in X$ is said to be a fixed point of multivalued mapping T if $x \in Tx$. We denote the set of fixed points of T by Fix(T).

Let (X, d) be a metric space. Let CB(X) denote the collection of non empty closed bounded subsets of X. For A, B $\in$ CB(X) and x $\in$ X, define

$$d(x, A) = \inf_{a \in A} d(x, a)$$

$$\text{and } H(A, B) = \max\left\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\right\}.$$

Note that H is called the Hausdorff metric induced by the metric d.

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Definition 1.2[11]: Let X be a nonempty set. A function $p : X \times X \longrightarrow R^+$ is said to be a partial metric on X if for any $x, y, z \in X$, the following conditions hold:

(i)$p(x, x) = p(y, y) = p(x, y)$ if and only if $x = y$,
(ii)$p(x, x) \leq p(x, y)$,
(iii)$p(x, y) = p(y, x)$,
(iv)$p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

The pair (X, p) is then called a partial metric space.
If $p(x, y) = 0$, then (i) and (ii) imply that $x = y$. But the converse does not hold always. A trivial example of a partial metric space is the pair (R^+, p), where $p : R^+ \times R^+ \longrightarrow R^+$ is defined as $p(x, y) = \max\{x, y\}$.

Example 1.3[11]: If $X = \{[a, b]: a, b \in R, a \leq b\}$, then

$$p([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\}$$

defines a partial metric p on X.

Each partial metric p on X generates a T_0 topology τ_p on X which has a base the family open p- balls $\{B_p(x, \varepsilon): x \in X, \varepsilon > 0\}$, where

$$B_p(x, \varepsilon) = \{y \in X: p(x, y) < p(x, x) + \varepsilon\}, \text{ for all } x \in X \text{ and } \varepsilon > 0.$$

Observe that a sequence $\{x_n\}$ in a partial metric space (X, p) converges to a point $x \in X$, with respect to τ_p, if and only if $p(x, x) = \lim_{n\to+\infty} p(x, x_n)$.

If p is a partial metric on X, then the function $p^s : X \times X \longrightarrow R^+$ given by

$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$, defines a metric on X.

Definition 1.4[11]: Let (X, p) be a partial metric space.

(a)A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if $\lim_{n, m\to+\infty} p(x_n, x_m)$ exists and is finite.

(b)(X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p to a point $x \in X$ such that $\lim_{n\to+\infty} p(x_n, x) = p(x, x)$. In this case, we say that the partial metric p is complete.

Lemma 1.5[4, 11]: Let (X, p) be a partial metric space. Then:

(a) A sequence $\{x_n\}$ in X is said to be a Cauchy sequence (X, p) if and only if it is a Cauchy sequence in metric space (X, p^s).
(b) A partial metric space (X, p) is complete if and only if the metric space (X, p^s) is complete. Moreover, $\lim_{n\to+\infty} p^s(x_n, x) = 0$ iff $\lim_{n,m\to+\infty} p(x_n, x_m) = \lim_{n\to+\infty} p(x_n, x) = p(x, x)$.

In 2012, Aydi et al. [3] defined a partial Hausdorff metric as follows:

Let (X, p) be a partial metric space. Let $CB^P(X)$ be the family of all nonempty, closed and bounded subsets of the partial metric space (X, p), induced by the partial metric p. Note that closedness is take from (X, τ_p), τ_p is the topology induced by p and boundedness is given as follows:

A is a bounded subset in (X, p) if there exist $x_0 \in X$ and $M \geq 0$ such that for all $a \in A$, we have $a \in B_P(x_o, M)$, that is, $p(x_0, a) < p(a, a) + M$. For $A, B \in CB^P(X)$ and $x \in X$, define

$p(x, A) = \inf\{p(x, a), a \in A\}$,
$\delta_P(A, B) = \sup\{ p(a, B): a \in A\}$
and $\delta_P(B, A) = \sup\{ p(b, A): b \in B\}$.

The mapping $H_p: CB^p(X) \times CB^p(X) \rightarrow [0, +\infty)$ define by

$$H_p(A, B) = \max\{ \delta_p(A, B), \delta_p(B, A)\}$$

is called a partial Hausdorff metric induced by p.

It is immediate to check that $p(x, A) = 0 \Longrightarrow p^s(x, A) = 0$,

where $p^s(x, A) = \inf\{p^s(x, a), a \in A\}$.

Remark 1.6[4]: Let (X, p) be a partial metric space and A any nonempty set in (X, p), then $a \in \bar{A}$ if and only if $p(a, A) = p(a, a)$, where $\bar{A}$ denotes the closure of A with respect to the partial metric p. Note that A is closed in (X, p) if and only if $A = \bar{A}$.

Now, we shall study some properties of mapping

$\delta_p : CB^p(X) \times CB^p(X) \to [0, +\infty)$.

Proposition 1.7[3]: Let (X, p) be a partial metric space. For any
$A, B, C \in CB^p(X)$ we have the following:

(i) $\delta_p(A, A) = \sup\{ p(a, a): a \in A\}$;

(ii) $\delta_p(A, A) \leq \delta_p(A, B)$;

(iii) $\delta_p(A, B) = 0$ implies that $A \subseteq B$;

(iv) $\delta_p(A, B) \leq \delta_p(A, C) + \delta_p(C, B) - \inf\limits_{c \in C} p(c, c)$.

Proposition 1.8[3]: Let (X, p) be a partial metric space. For any $A, B \in CB^p(X)$,
we have the following:

(i)$H_p(A, A) \leq H_p(A, B)$;
(ii)$H_p(A, B) = H_p(B, A)$;
(v) $H_p(A, B) \leq H_p(A, C) + H_p(C, B) - \inf\limits_{c \in C} p(c, c)$.

Corollary 1.9[3]: Let (X, p) be a partial metric space. For any $A, B \in CB^P(X)$, the following holds

$H_p(A, B) = 0$ implies that $A = B$.

Remark 1.10[3]: The converse of Corollary 1.9 is not true in general as it is clear from the following example.
Example 1.11[3]: Let X = [0, 1] be endowed with the partial metric

$p : X \times X \longrightarrow R^+$ defined by $p(x, y) = \max\{x, y\}$. From (i) of proposition 1.7, we have

$H_p(X, X) = \delta_p(X, X) = \sup\{x : 0 \leq x \leq 1\} = 1 \neq 0$.

Remark 1.12[3]: It is easy to show that any Hausdorff metric is a partial metric, but the converse is not true (see example 1.11).
Definition 1.13[1]: An element $x \in X$ is said to be a coincidence point of
$T : X \longrightarrow CB^p(X)$ and $f : X \longrightarrow X$ if $fx \in Tx$. We denote
$C(f, T) = \{ x \in X : fx \in Tx\}$,
the set of coincidence point of T and f.

Definition 1.14[6]: Mappings $f : X \longrightarrow X$ and $T : X \longrightarrow CB^p(X)$ are weakly compatible if they commute at their coincidence points, that is, if $f(Tx) = T(fx)$ whenever $fx \in Tx$.

Definition 1.15[6]: Let $T : X \longrightarrow CB^p(X)$ be a multivalued mapping and $f : X \longrightarrow X$ be a single valued mapping. The mapping f is said to be T-weakly commuting at $x \in X$ if $ffx \in Tfx$.

Definition 1.16[1]: An element $x \in X$ is a common fixed point of $T, S : X \longrightarrow CB^p(X)$ and $f : X \longrightarrow X$ if $x = fx \in Tx \cap Sx$.

MAIN RESULTS

Lemma 2.1[3]: Let (X, p) be a partial metric space, $A, B \in CB^p(X)$. Suppose that $\varepsilon > 0$ and $H_p(A, B) < \varepsilon$. Then for all $a \in A$ there exist $b \in B$ such that $p(a, b) < \varepsilon$.

Theorem 2.2: Let (X, p) be a complete partial metric space. Let $T, S : X \longrightarrow CB^p(X)$ are two multivalued mappings and $f : X \longrightarrow X$ a pair of single valued mapping satisfying the following

(i)$S(X) \subseteq f(X)$ and $T(X) \subseteq f(X)$,
(ii)$f(X)$ is complete.
(iii)there exist a mapping $\varphi : (0, \infty) \longrightarrow [0, 1)$ such that

$$\limsup_{r\to t^+} \varphi(r) < 1 \text{ for all } t \in [0, \infty), \tag{1}$$

and for all $x, y \in X$,

$$H_p(Sx, Ty) \le \varphi(p(fx, fy))\, p(fx, fy). \tag{2}$$

Then T, S and f have a coincidence point in X. That is, there exist $y \in X$ such that $fp \in Sp \cap Tp$.

Proof :- Let $x_0 \in X$ and let $x_1 \in X$ be such that $fx_1 \in Sx_0$. From (2) we have $p(fx_1, Tx_1) \le H_p(Sx_0, Tx_1) \le \varphi(p(fx_0, fx_1))\, p(fx_0, fx_1) < p(fx_0, fx_1)$.

By Lemma 2.1 and (i). We can take $x_2 \in X$ such that $fx_2 \in Tx_1$ and $p(fx_1, fx_2) < p(fx_0, fx_1)$. From (2) we have

$$p(fx_1, Sx_2) \le H_p(Sx_2, Tx_1) \le \varphi(p(fx_1, fx_2))\, p(fx_1, fx_2) < p(fx_1, fx_2).$$

Again, by Lemma 2.1 and (i), we can take $x_3 \in X$ such that

$fx_3 \in Sx_2$ and $p(fx_2, fx_3) < p(fx_1, fx_2)$.

Continuing this process, we can construct a sequences $\{x_n\}$ in X such that for $n = 0, 1, 2...$

$fx_{2n+1} \in Sx_{2n}$, $fx_{2n+2} \in Tx_{2n+1}$,
$p(fx_{n+1}, fx_{n+2}) < p(fx_n, fx_{n+1})$

Since $\{p(fx_n, fx_{n+1})\}$ is a nonincreasing sequence in $[0, \infty)$, the sequence $\{p(fx_n, fx_{n+1})\}$ is convergent. From (1) there exist $r \in (0, 1)$ such that

$\limsup_{r\to t^+} \varphi(p(fx_n, fx_{n+1})) = r$. Thus for any $k \in (r, 1)$, there exist $N \in N$ such that for all $n \ge N$. $\varphi(p(fx_{n-1}, fx_n)) < k$. Hence we have for $n \ge N$,

$$p(fx_n, fx_{n+1}) \le \varphi(p(fx_{n-1}, fx_n))\, p(fx_{n-1}, fx_n)$$
$$< k\, p(fx_{n-1}, fx_n).$$

Thus for each $m > n \ge N$, we have

$$p(fx_n, fx_m) \le [p(fx_n, fx_{n+1}) + p(fx_{n+1}, fx_{n+2}) + \ldots + p(fx_{m-1}, fx_m)] -$$
$$[p(fx_{n+1}, fx_{n+1}) + p(fx_{n+2}, fx_{n+2}) + \ldots + p(fx_{m-1}, fx_{m-1})]$$
$$\le [k^{n-N} + k^{n-N+1} + k^{m-N-1}]\, p(fx_N, fx_{N+1})$$
$$\le \frac{k^{n-N}}{1-k}\, p(fx_N, fx_{N+1}).$$

which implies $\lim_{m,n\to\infty} p(fx_n, fx_m) = 0.$ (3)

By the definition of p^s, we have

$$p^s(fx_n, fx_m) = 2p(fx_n, fx_m) - p(fx_n, fx_n) - p(fx_m, fx_m)$$
$$\le 2p(fx_n, fx_m)$$

Applying (3), this yields

$$\lim_{m,n\to\infty} p^s(fx_n, fx_m) = 0. \tag{4}$$

hence $\{fx_n\}$ is a Cauchy sequence in (X, p^s). Since (X, p) is complete, then from lemma (1.5), (X, p^s) is a complete metric space. By the completeness of f(X), we have $\{fx_n\}$ converges to some $u \in X$ with respect to the metric p^s. Therefore there exists $q \in X$ such that u = fq. That is

$$\lim_{n\to+\infty} fx_n = fq. \tag{5}$$

Then by lemma 1.5 and (5), we obtain that

$$p(fq, fq) = \lim_{n\to+\infty} p(fx_n, fq) = \lim_{n,m\to+\infty} p(fx_n, fx_m) \tag{6}$$

From (3) and (6), we can conclude that p(fq, fq) = 0.Therefore, for each $n \in N$, From (2) we have

$$\begin{aligned} p(fx_{n+2}, Sq) &\leq H_p(Sq, Tx_{2n+1}) \\ &\leq \varphi(p(fq, fx_{2n+1}))\, p(fx_q, fx_{2n+1}) \\ &< p(fx_q, fx_{2n+1}). \end{aligned}$$

Letting $n \to \infty$, we have p(fq, Sq) = 0, we have $fq \in Sq$. Similary, we can show $fq \in Tq$. Therefore, we have $fq \in Tq \cap Sq$.

Example 2.3: Let X = [4, ∞) and p(x, y) = max{x, y} for all $x, y \in X$. Then p is a partial metric space on X. Define S, T, f and g on X as follows:

Sx = x + 60, Tx = x + 56, $fx = x^3$ and $gx = x^2$.

Then,
$p(fx, gy) = \max\{x^3, y^2\} = x^3$ and
$p(Sx, Ty) = \max\{ x + 60, y + 56\} = x + 60$
$p(Sx, Ty) = x + 60 \leq x^3 = p(fx, gy)$.
$p(Sx, Ty) \leq p(fx, gy)$.

Thus, (1) hold for all $x, y \in X$. Also the other hypotheses (i) and (ii) are satisfied. It is seen that S(4) = f(4) = 64 and T(8) = g(8) = 64.

Therefore, S and f have the coincidence at the point u = 4, T and g at the point w = 8, and S(4) = T(8).

Theorem 2.4: Let (X, p) be a complete partial metric space. Let

$T, S : X \longrightarrow CB^p(X)$ be multivalued mappings and $f : X \longrightarrow X$ be a single valued mapping satisfying , for each $x, y \in X$,

$$H_p(Sx, Ty) \leq ap(fx, fy) + b[p(fx, Sx) + p(fy, Ty)] + c[p(fx, Ty) + p(fy, Sx)]$$

where $a, b, c \geq 0$ and $0 < a + 2b + 2c < 1$. If f X is a closed subset of X and $TX \bigcup SX \subseteq f X$, then f, T and S have a coincidence in X. Moreover, if f is both T-weakly commuting and S-weakly commuting at each $z \in C(f, T)$, and ffz = fz, then f, T and S have a common fixed point in x.

Proof :- If f = g in Theorem 2.2, we obtain that there exist points u and w in X such that $fu \in Su$, $fw \in Tw$, fu = fw and Su = Tw.

As $u \in C(f, T)$, f is T-weakly commuting at u and ffu = fu. Set v = fu. Then we have fv = v and $v = ffu \in T(fu) = Tv$. Now, since also $u \in C(f, S)$ then f is S-weakly commuting at u and so we obtain $v = fv = ffu \in S(fu) = Sv$. Thus we prove that $v = fv \in Tv \cap Sv$, that is v is a common fixed point of f, T and S.
If f = g = Ix (Ix be the identity mapping on X) in Theorem 2.2, then we obtain the following common fixed point result.

Corollary 2.6 : Let (X, p) be a complete partial metric space.

Let $T, S : X \longrightarrow CB^p(X)$ be multivalued mappings satisfying , for each $x, y \in X$,
$H_p(Sx, Ty) \leq ap(x, y) + b[p(x, Sx) + p(y, Ty)] + c[p(x, Ty) + p(y, Sx)]$

where $a, b, c \geq 0$ and $0 < a + 2b +2c < 1$. Then there exists a point z in X, such that $z \in Tz \cap Sz$ and Sz = Tz.

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