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GENERALIZATION OF ASPLUND INEQUALITIES ON LIPSCHITZ FUNCTIONS IN FUZZY NORMED LENEAR SPACES

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ABSTRACT

In this paper is to generalised the Asplund inequalities on lipschitz functions in Fuzzy normed linear spaces. 1991 AMS SUBJECT CLASSIFICATION CODES: 49J15, secondary 49J20, 93c15, 93c20.

INTRODUCTION

Generalization of Asplund inequalities on lipschitz functions were considered by ROLEWICZ (Rolewicz, 1993).The principal objective of this paper is to generalised the idea on lipschitz functions in fuzzy linear spaces.

2.Some Fundamental Definitions and Theorems

Definition2. 1 (Samanta, 2014). Let X be any non empty set and F(X) be the set of all Fuzzy sets on X . For U,V $\in$ F(X)

and k $\in$ K the field of real numbers, define U+V={(x + y, λ $\wedge$ μ)|(x, λ) $\in$ U,(y, μ) $\in$ V} and

k U ={(k x , λ)|(x, λ) $\in$ U}.

Definition2. 2 (Samanta, 2014). Let X be a linear space over K (field of real or complex numbers). Then a fuzzy linear space $\widetilde{X}$ =X $\times$(0, 1] over the field K where the addition on $\widetilde{X}$ are defined by (x , λ)+(y,μ)=(x + y , λ $\wedge$ μ)
k(x ,λ)=(k x ,λ) is a fuzzy normed linear space if to every (x ,λ) $\in$ $\widetilde{X}$ there is corresponds a non-negative real number || (x ,λ)|| ,called fuzzy normed of
(x , λ) in such a way that

|| (x , λ) ||=0 iff x=0 the zero element of X, λ $\in$ (0, 1],

|| k(x, λ)||=| k | ||(x, λ)|| for (x, λ) $\in$ $\widetilde{X}$ and all k $\in$ K,

|| (x , λ)+(y,μ)||≤||(x ,λ $\wedge$ μ)||+||(y ,λ $\wedge$ μ)|| for all (x ,λ),(y,μ) $\in$ $\widetilde{X}$,

|| (x , $\vee$ t λt)||= $\wedge_t$ ||(x , λt)|| for λt $\in$ (0,1].

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Definition 2. 3 (Samanta, 2014). Let X be a linear space over the field of real or complex numbers say K, then a fuzzy subset N of $X \times R$ (R is the set of real numbers) is called a fuzzy norm on X if and only if for all

x ,u $\in$ X and c $\in$ K,
(N1) for all t $\in$ R with t<0, N(x , t)=0,
(N2) for all t $\in$ R with t >0, N (x ,t)=1 if and only if x=0.
(N3) for all t $\in$ R with t >0, N (cx,t)=N(x ,t/|c|),if c≠0.
(N4) for all s ,t $\in$ R , x , u $\in$ X , N (x + u ,s + t) ≥ min{N(x , s), N(u , t)}
(N5) N(x,·) is a non-decreasing function of R and lim N (x ,t)=1,then the pair (X ,N) is called fuzzy normed linear space.

Definition2.4 (Samanta, 2014). Let (X,N) be a fuzzy normed linear space .we define || x || α = inf {t : N(x , t) ≥ α}, ~ $\in$ (0,1).

Then {|| · || α: ~ $\in$ (0,1)} is an ascending family of norms on X (or) α-norms on X corresponding to the fuzzy norm on X.

Definition2.5 (Samanta, 2014). Let X be a non-empty set and F(X) be the set of all fuzzy sets on X. If f $\in$ F(X), then f = {(x,µ)/x $\in$ X and µ $\in$ (0,1]}. Clearly f is a bounded function for |f(x)| ≤1. Let K be the space of real numbers then F(X) is a linear space over the field K where the addition and scalar multiplication are defined by

f + g = { (x,µ)+(y ,ŋ) }={(x + y ,µ $\wedge$ ŋ) / (x,µ) $\in$ f and (y,ŋ) $\in$ g }

And

K f= (k f,µ) / (x , µ) $\in$ f},where k $\in$ K.

The linear space F(X) is said to be normed space if for every f $\in$ F(X), there is associated a non-negative real number || f || called the norm of f in such a way that

|| f || = 0 if and only if f = 0. For || f || = 0 if and only if { $\|(x,«)\| : (x,«) \in f\} = 0$,x = 0 ,µ $\in$ (0,1] if and only if f = 0.
|| k f || =| k ||| f ||, k $\in$ K. For || k f|| = {|| k (x, µ) || / (x, µ)$\in$ f, k $\in$ K } = {| k ||| x , µ || / (x ,µ) $\in$ f } = | k ||| f ||.
|| f + g || ≤ || f ||+ || g || for every f , g $\in$ F(X).

For || f + g || ={|| (x,µ) + (y ,ŋ) || / x , y $\in$ X , µ,ŋ $\in$ (0,1]}
={ || (x + y) , (µ ŋ) || / x , y $\in$ X, µ,ŋ $\in$ (0,1]}
≤ {|| (x,µ $\wedge$ ŋ)|| + || (y,µ $\wedge$ ŋ)|| / (x, µ) $\in$ f and (y ,ŋ) $\in$ g }= || f || + || g ||.
Then (F(X), || · ||) is a normed linear space.

Definition 2.6 (Samanta, 2014): A 2-fuzzy set on X is a fuzzy set on F(X).

Definition2.7 (Samanta, 2014). Let F(X) be a linear space over the real field K. A fuzzy subset N of $F(X) \times F(X) \times R$. (R , the set of real numbers) is called a 2-fuzzy 2-norm on X (or fuzzy 2-norm on F(X)) if and only if,
(N1) for all t $\in$ R with t≤0 , N (f1 ,f2 ,t)=0 ,
(N2) for all t $\in$ R with t ≥ 0 ,N (f1 ,f2 ,t)=1 , if and only if f1 and f2 are linearly dependent,
(N3) N (f1 ,f2 ,t) is invariant under any permutation of f1 and f2 ,
(N4) for all t $\in$ R , with t ≥ 0 , N (f1 ,cf2 ,t)= N (f1 ,f2 ,t / |c|) if c ≠ 0, c $\in$ K (field),
(N5) for all s ,t $\in$ R , N(f1 , f2+f3 , s + t) ≥ min { N (f1 ,f2 ,s), N (f1 ,f3 , t)},
(N6) N (f1 ,f2 ,.) : (0,∞) $\rightarrow$ [0,1] is continuous,
(N7) $\lim_{t\to\infty}$ N (f1 ,f2 ,t)=1.

Then (F(X),N) is a fuzzy 2-normed linear space or (X,N) is a 2-fuzzy 2-normed linear space .

Let X be a linear space. Then $(\tilde{X}, \|.\|)$ is a fuzzy normed linear space. Let $\tilde{X}^*$ denote the class of all continuous convex function defined on X.
Definition2. 8. Let F(X) be the set of all maps from X to I,where I = [0,1]. Then λ is a fuzzy point in X , x its support and α its value , λ may be denoted by $a^{\sim}_{x}$.
Let γ $\in$ F(x). then $a^{\sim o}_{xo}$ $\in$ γ iff γ(x0) >α0,
Fuzzy point $a^{\sim}_{x}$, $a^{\sim\prime}_{x'}$ will be denoted by $a, a_i, i = 0,1,2,.....a$ respectively.

Let X be a reference set, I = [0,1] and let F(~) be the set of all maps from X to I. The partial order in F(X) is defined as follows $¥ < «$, where $¥, « \in F(X)$ if and only if $¥(x) \le «(x)\, for\ all\ \ x \in X$. Then (F(X),<) is a complete lattice. The suprema and infima are denoted by $\vee, \wedge$ respectively.

Definition2.9 (Samanta, 2014):

$$(1)\ «\ is\ a\ \ fuzzy\ set\ in\ X \Leftrightarrow « \in F(X),$$

$$(2)\ «_1 < «_2 \Rightarrow «_1(x) \le «_2(x),\ \forall\ x \in X.$$

$$(3)\ (\underset{\sim}{\vee} «_{\sim})(x) = \underset{\sim}{\sup}\{«_{\sim}(x)\}, \forall\ x \in X.$$

$$(4)\ (\underset{\sim}{\wedge} «_{\sim})(x) = \underset{\sim}{\inf}\{«_{\sim}(x)\}, \forall\ x \in X.$$

$$(5)\ «^c(x) = 1 - «(x), \forall\ x \in X.$$

$$(6)\ \tilde{1}(x) = 1,\ \tilde{0}(x) = 0, \forall\ x \in X.$$

Definition 2.10 (Samanta, 2014): Let $f : X \to Y, « \in F(X), ¥ \in F(Y)$

$$f(«)(y) = \sup_{x \in f^{-1}(y)} \{«(x)\},\ f(y) \ne ¤$$

$$= 0, \qquad f^{-1}(y) = ¤$$

$$f^{-1}(¥)(x) = ¥(f(x)), \forall\ x \in X.$$

Definition 2.11 (Samanta, 2014): S $\subset F(X)$ satisfy the following conditions:

(1) $\tilde{I} \in S, \tilde{O} \in S.$

(2) $« \in S, ¥ \in S \Rightarrow « \wedge ¥ \in S$.

(3) $«_{\sim} \in S \Rightarrow \underset{\sim}{\vee} «_{\sim} \in S$.

Then S is called a fuzzy topology on X and (X,S) a fuzzy topological space.

Definition 2.12 (Samanta, 2014): A fuzzy set $ª \in F(X)\ such\ that \sim \in (0,1)$ and

$$ª(x^1) = \sim,\ \ if\ \ x^1 = x$$

$$ª(x^1) = 0,\ \ if\ \ x^1 \ne x$$

is called a fuzzy point in X (written $ª)$, $x\ its\ \sup port\ and \sim its\ value$, $ª\ may\ be\ denoted\ by\ ª_x^{\sim}.$

Definition2.13 (Samanta, 2014): $ª_{x_0}^{\sim_0}\ is\ said\ to\ belong\ to\ ¥\ (written\ ª_{x_0}^{\sim_0} \in ¥)\ if\ \ and\ only\ if\ ¥(x_0) > \sim_0.$

Definition 2.14 [9]. Let $« \in F(X).$ Then $«$ is called a neighbourhood of p if $\exists ¬ \in S$ such that $p \in ¬ < «$ where $S \subset F(X).$

Let N_p denote the system of all neighbourhoods of p.If $« \in F(x), then\ « = \underset{x \in X}{V} ª_x\ where\ ª_x$ is a fuzzy point. From now on fuzzy point

$ª_x^{\sim}, ª_{x'}^{\sim'}, i = 1,2,\ldots\ldots\ldots, ª_{x'}^{\sim'}, will$ be denoted by $ª, ª_{i,} i = 0,1,\ldots\ldots ª$ respectively.

Definition2.15 (Samanta, 2014). A fuzzy linear space $\tilde{X} = X \times (0,1]$ over the number field K where the addition and scalar multiplication on $\tilde{X}$ are defined by

$$(x, ª) + (y, «) = (x + y, ª \wedge «),$$

$$©(x, y) = (©x, ª)$$

Then $\tilde{X}$ is a fuzzy linear space if to every $(x, ª) \in \tilde{X}$ there corresponds a non-negative real number, ||(x, λ)||, called the fuzzy normed of (x, λ) , such that

(1) $\| (x, ª) \| = 0 \; iff \; x = 0 \; the \; zero \; element \; of \; X, ª \in (0,1],$

(2) $\| ©(x, ª) \| = | © | \, \| (x, ª) \| \; \forall (x, ª) \in \tilde{X} \; and \; all \; © \in K,$

(3) $\| (x, ª) + (y, «) \| \le \| (x, ª \wedge «) \| + \| (y, ª \wedge «) \| \; \forall (x, ª), (y, «) \in \tilde{X},$

(4) $\| (x, \vee_t ª_t) \| = \wedge_t \| (x, ª_t) \| \; for \; ª_t \in (0,1],$

then $\tilde{X}$ will be defined as fuzzy normed linear space.

Definition 2.16 (Samanta, 2014). Let X be a any non-empty set and F(X) be the set of all fuzzy sets on X. For U,V $\in$ F(X) and k $\in$ K the field of real numbers , define

$$U, V = \{(x = y, ª \wedge «) \mid (x, ª) \in U, (y, «) \in V\},$$

$$and \; ©U = \{(©x, ª) \mid (x, ª) \in U\}$$

Definition 2.17 (Samanta, 2014). The fuzzy subset $«_1 + «_2 is \; defined \; by$

$(«_1 + «_2)(x) = \vee \{«_1(x) + «_2(x) : x = x_1 + x_2\}.$

And for a scalar t of K and a fuzzy subset $« \; of \; X,$ the fuzzy subset $t«$ is defined by

$$(t«)(x) = \begin{cases} «(x/t) & if \; t \ne 0 \\ 0 & if \; t = 0 \; and \; x \ne 0 \\ \vee \{«(y) \mid y \in X\} & if \; t = 0 \; and \; x = 0 \end{cases}$$

Definition 2.18 (Samanta, 2014). $f \in I^X$ is said to be

convex if $tf + (1-t) f \subseteq f \;\; for \; each \; t \in (0,1]$

balanced if $tf \subseteq f \;\; for \; each \; t \in K \; with \, |t| \le 1$

absorbing if $\vee \{tf(x) \mid t > 0\} = 1$ for all x $\in$ X.

Let X be a linear space . Then $\left(\tilde{X}, \| \; . \; \|\right)$ is a fuzzy normed linear space. Let $\tilde{X}^*$ denote the class of all continuous convex function defined on $\tilde{X}$. Let f(x) be a lower-semicontinuous function defined on $\tilde{X}$ we denotes

$$f^*\left(x^*\right) = \sup\{x^*(x) - f(x)\} \quad \text{.......... (1)}$$

where $f^*\left(x^*\right)$ will be called Fenchel dual function $, x^* \in \tilde{X}^*.$

Theorem 2.1: (Asplund, 1968 and Bronstedt, 1964). Let f(x) be a lower-semi- continuous convex function defined on a banach space (X,‖·‖). Let γ be a convex function mapping the interval [0,+∞) into [0, +∞] such that γ(0) =0 . For a fixed $x_0 \in$ X and $x_0^* \in X^*$ the following inequality are equivalent

$$f(x) - f(x_0) \ge x_o^*(x - x_o) + ¥(\| x - x_0 \|) \;\; for \; all \; x \in X \quad \text{.......... (2)}$$

$$f^*(x^*) - f^*(x_0^*) \le (x^* - x_0^*)(x_0) + ¥^*(\| x^* - x_0^* \|^*) \;\; for \; all \; x^* \in X^* \quad \text{.......... (3)}$$

where $¥^*$ denote the function conjugate to γ, $¥^*(t) = \sup_{u \succ 0}[ut - ¥u]$ .... (4)

In the present it will be shown that theorem 2.1 holds on fuzzy space.

Now we shall establish the following results.

(1) $f^*(¨) \ge g^*(¨) \; iff \;\; f(x) \le g(x), \; for \; all \; x \in \tilde{X} \; and ¨ \in ¤,¤$ is a certain family of functions defined on $\tilde{X}$.

Proof. By (1)

$$f^*(¨) \ge g^*(¨)$$

$$\Leftrightarrow \sup_{x \in \tilde{X}}[¨(x) - f(x)] \ge \sup_{x \in \tilde{X}}[¨(x) - g(x)]$$

$$\Leftrightarrow f(x) \le g(x)$$

Similarly,

(2) $f^*(¨ + r) = f^*(¨) + r$ for all r $\in$ R,

(3) $(f^* + r)^¨ = f^*(¨) - r$ for all r$\in$ R ,

(4) $f(x) + f^*(¨) \geq ¨(x)$.

We say that a function f is ¤ -convex if it is a majorant of the function of¤ ,

$$f(x) = \sup\{¨(x) : ¨(x) \in ¤, ¨ \leq f\} \quad \text{.......} (5)$$

Observe that the space $\tilde{X}$ induces on the family ¤ also family of functions by formula

$$x(¨) = ¨(x).$$

Thus for functions defined on ¤ we can speak about $\tilde{X}$ -convexity. It is easy to see that the function $f^*(¨)$ is always $\tilde{X}$ -convex. We say that function $¨_0 \in ¤$ is a ¤ -subgradient of the function f(x) at a point x_0 if

$$f(x) - f(x_o) \geq ¨_0(x) - ¨_0(x_0) \text{ for all } x \in \tilde{X}. \quad \text{.......} (6)$$

It is easy to see that a function $¨_0$ *is a* ¤ -subgradient of a function f(x) at the point x_0 , then x_0 is a $\tilde{X}$ -subgradient of a function f^* at the point x_0 .

It is easy to see that a function $¨_0$ *is a* ¤ -subgradient of a function f at the point x_0 , then $x_0$is a $\tilde{X}$ -subgradient of a function of a function f^* at the point x_0 .

Definition 2. 19 : Let $\tilde{X}$ be the set of all maps from $\tilde{X}$ to I . Then λ is a fuzzy point in $\tilde{X}$, x its support and α its value , λ may be denoted by $ª_x^{~}$. Let γ ¤ $\in$ F(x) . then $ª_{x_0}^{~_0} \in$ γ iff γ(x_0)>$α_0$.

Fuzzy point $ª_x^{~}$, $ª_{x'}^{~'}$ *will be denoted by* $ª, ª_i$, $i = 0,1,2,......ª$ respectively.

Definition 2. 20: Let ρ be set of all fuzzy points in $\tilde{X}$. Then fuzzy metric space is defined as follows: if a map d: ρxρ $\longrightarrow$[0, ∞],
Satisfying

- $d(ª_1, ª_2) = 0 \Leftrightarrow x_1 = x_2 \text{ and } ~_1 \leq ~_2$
- $d(ª_1, ª_2) = d(ª_2, ª_1)$
- $d(ª_1, ª_3) \leq d(ª_1, ª_2) + d(ª_2, ª_3)$
- $d(ª_1, ª_2) = r$ *where* $r > 0 \Rightarrow$ *there exist* $~' > ~_1$ *such that* $d(ª_{x_1}^{~'}, ª_2) < r$, is called a fuzzy matric space.

Definition 2. 21: A sequence { $λ_n$ } of fuzzy points converges to λ iff for each μ $\in$ $N_λ$ (i.e.the system of all neighbourhoods of fuzzy point p in $\tilde{X}$), there exist positive integer N such that n ≥ N $\Rightarrow ª_n \in «$ (*written* $ª_n \longrightarrow ª$).

Definition 2.22: A sequence { $λ_n$ } is called a Cauchy sequence if

$$\lim_{n,m\to\infty} d(ª_n, ª_m) = 0$$

Definition 2.23: A fuzzy metric space $(\tilde{X}, d)$ is said to be complete iff every Cauchy sequence in $(\tilde{X}, d)$ converges with respect to the fuzzy metric.

Definition 2.24: Two fuzzy metric space $(\tilde{X}, d)$ and $(\tilde{X}', d')$ are isometric if there exits a one-to-one mapping ¨ from $(\tilde{X}, d)$ onto $(\tilde{X}', d')$ such that for every $ª_1, ª_2 \ in\ (\tilde{X}, d) \Rightarrow d(ª_1, ª_2) = d(¨(ª_1), ¨(ª_2))$ .The mapping ¨ is called an isomery.

Definition 2. 25: A complete fuzzy metric space (N^* ,d^*) is a completion of $(\tilde{X}, d)$ if $(\tilde{X}, d)$ is isometric to a dense fuzzy subset of (X^*, d^*) .

Let $(\tilde{X}, d)$ *be a metric space. Let* ¤ *be a* subclass of the space of all lipschitzian functions defined on $\tilde{X}$. *Let* $d_l(¨_1, ¨_2) = \sup_{x_1, x_2 \in \tilde{X}} \{\frac{[¨_1(x_1) - ¨_2(x_2)] - [¨_1(x_2) - ¨_2(x_2)]}{d_x(x_1, x_2)} : x_1 \neq x_2\}$

Then obviously d_L is quasimetric, i.e. it is symmetric and satisfies the triangle inequality .If $d_L(¨_1, ¨_2)$=0,then the difference of $¨_1$ *and* $¨_2$ is a constant function, i.e. $¨_1(x) = ¨_2(x) + c$.

Now we consider the quotient space ¤ /R, where we identify all functions , which differ on constants. It is easy to see that d_L is a metric on ¤ /R .

Theorem 2.2: Let f(x) be a ¤ -convex function. Suppose that

$$f^*(¨) \geq f^*(¨_0) + ¨(x_0) - ¨_0(x_0) + ¥(d_L(¨, ¨_0)) \quad (7)$$

holds.Then

$$f(x) \leq f(x_0) + ¨_0(x) - ¨_0(x_0) + ¥^*(d(x, x_0)) \quad (8)$$

where $¥^*(t)$ *is a dual function to the function* $¥, ¥^*(t) = \sup_{u \succ 0}[ut - ¥(u)]$.

Proof: Since the function f(x) is ¤ -convex for each x $\in \tilde{X}$ and each ε > 0 there is a ¨ $\in$ ¤ such that

$$f(x) + f^*(¨) - ¨(x) < £.$$

Thus

$$¨(x) - f(x) - ¨(x_0) + £ \geq ¨_0(x_0) - f(x_0) - ¨_0(x_0) + ¥(d_L(¨, ¨_0) \quad (9)$$

and

$$f(x) \leq f(x_0) + ¨(x) - ¨(x_0) - ¥(d_L(¨, ¨_0)) + £$$
$$= f(x_0) + ¨_0(x) - ¨_0(x_0) + [¨(x) - ¨(x_0)] - [¨_0(x) - ¨_0(x_0)] - ¥(d_L(¨, ¨_0)) + £$$
$$\leq f(x_0) + ¨_0(x) - ¨_0(x_0) + d_L(¨, ¨_0) d(x, x_0) - ¥(d_L(¨, ¨_0)) + £$$
$$\leq f(x_0) + ¨_0(x) - ¨_0(x_0) + \sup_{t>0}[t d(x, x_0) - ¥(t)] + £$$
$$= f(x_0) + ¨_0(x) - ¨_0(x_0) + ¥^*(d(x, x_0)) + £$$

The arbitraryness of £ implies (8).

Let $(\tilde{X}, d)$ be a metric space . Let ¤ consist of Lipschitzian functions . As it was shown before the metric d induces on the space ¤ /R a metric d_L . Observe that $\tilde{X}$ can be interpreted as a set of Lipschitzian functions of (¤ /R ,d_L). Then we can consider on the space $\tilde{X}$ a corresponding Lipschitzian metric which we denote as $d_L(d_L(x, y))$.

Using this metric we can obtain the following proposition

Theorem 2.3: suppose that

$$f(x) \geq f(x_0) + ¨_0(x) - ¨_0(x_0) + ¥(d_L(d_L(x, x_0))) \quad (10)$$

Then

$$f^*(¨) \leq f^*(¨_0) + ¨(x_0) - ¨_0(x_0) + ¥^*(d_L(¨, ¨_0)) \quad (11)$$

In the case when the metric $d_L(d_L(x,y))$ coincides with the initial metric d(x ,y), $d_L(d_L(x,y))$ =d(x ,y), obtains a simpler from

$f(x)-¨_0(x)\geq f(x_0)-¨_0(x_0)+¥(d(x,x_0))$ …….(12)

Theorem 2.4: Let $(\tilde{X},d)$ be a metric space .Let ¤ denote a class of Lipschitzian functions defined on $\tilde{X}$, such that for each $x_0, ¨_0, ¨, t, ¢, £ > 0$ there is a x such that

$$|d(x,x_0)-t|<¢t \quad \ldots\ldots (13)$$

and

$$[¨(x)-¨(x_0)]-[¨_0(x)-¨_0(x_0)]-d_L(¨,¨_0)d(x,x_0)|<£ \quad \ldots\ldots (14)$$

Let $f(x)$ *be a* ¤ *– convex function. If* $¨_0$ *is a* ¤ *– subgradient of the function* $f(x)$ *at a point* x_0 *and*

$$f(x)\leq f(x_0)+¨_0(x)-¨_0(x_0)+¥^*(d(x,x_0)), \quad \ldots\ldots (15)$$

then

$$f^*(¨)-¨(x_0)\geq f^*(¨_0)-¨_0(x\ )+¥(d_L(¨,¨_0)) \ldots\ldots (16)$$

Proof: Applying Fehchel-Moreau inequality to (10) and using the fact for $¨_0$ we have equality we obtain

$$¨(x)-f^*(¨)\leq -f^*(¨_0)+¨_0(x)+¥^*(d(x,x_0)) \ldots\ldots (17)$$

Then

$$f^*(¨)\geq f^*(¨_0)+¨(x)-¨_0(x)-¥^*(d(x,x_0)) \quad \ldots\ldots (18)$$

and

$$f^*(¨)\geq f^*(¨_0)+[¨(x_0)-¨_0(x_0)]+[¨(x)-¨(x_0)]-[¨_0(x)-¨_0(x_0)]-¥^*(d(x,x_0)) \quad \ldots\ldots (19)$$

By our assumptions (13) *and* (14) *we have*

$$f^*(¨)\geq f^*(¨_0)+[¨(x_0)-¨_0(x_0)]+\sup_{t>0}(d_L(¨,¨_0)(1-¢)t-¥^*((1+¢)t))-£$$

$$= f^*(¨_0)+[¨(x_0)-¨_0(x_0)]+¥(\frac{1-¢}{1+¢}d_L(¨,¨_0))-£ \ldots\ldots (20)$$

The continuity of γ and the arbitrarily of δ and ε implies (16).

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